INVARIANT Γ-INTEGRALS AND SOME OF THEIR APPLICATIONS IN MECHANICS

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A fundamental property of analytic functions of a complex variable is the invariance of the contour of an analytic function relative to the integration path. This property permits the integration contour to be deformed in the domain of analyticity without changing the value of the integral. It is of interest to seek similar invariant integrals in other problems of mathematical physics, not reducing to the planar problems of potential theory.

In this paper a system of energy integrals (Γ -integrals) has been constructed for a continuous medium with arbitrary rheological and electromagnetic properties (Sect. 1). Next, a general theory is proposed for the motion of the singularities based on the invariant Γ-integrals (Sect. 2). The following questions are examined as some special cases of this theory: breakdown of dielectrics by an electric field (motion of charges and currents, Sect. 3); drag of a body in ideal incompressible fluid flow (motion of dipoles, vortices and sinks in an ideal fluid, Sect. 4); motion of cracks and dislocations in elastic bodies (Sect. 5) etc.

1. Invariant Γ-integrals in the case of an electromagnetic deformable medium. Let us consider a deformable continuous medium located in an electromagnetic field in the most general case of interaction of field and medium (the electromagnetic field causes a deformation of the medium and, conversly, a deformation of the medium generates an electromagnetic field). The state of an electromagnetic deformable medium is characterized by the field vectors $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$, the displacement vector $\mathbf{u}$ and the stress and strain tensors σ_{ik} and ε_{ik}. The following equations hold:

Maxwell's equations (1.1)

$$\varepsilon_{ijk}E_{j,k} + \frac{\partial B_i}{\partial t} = 0, \quad \varepsilon_{ijk}H_{j,k} - \frac{\partial D_i}{\partial t} = J_i$$

$$D_{i,i} = \rho, \quad B_{i,i} = 0, \quad J_{i,i} + \frac{\partial \rho}{\partial t} = 0$$

Newton's equations (1.2)

$$\sigma_{ij,j} = \delta u_i^{\cdot\cdot}$$

kinematic conditions for small deformations

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i} \tag{1.3}$$

local energy conservation law

$$U^{\cdot} = q_{i,i} + E_i D_i^{\cdot} + H_i B_i^{\cdot} + \sigma_{ij}\varepsilon_{ij}^{\cdot} \tag{1.4}$$

Here **J** is the current density vector, ρ is the charge density, δ is the density of the medium, x_i are fixed rectangular Cartesian coordinates, t is time, **q** is the uncompensated heat flow vector, $U^{\cdot}$ is the rate of change of the internal energy of the medium in a unit volume; the dot denotes the total time derivative; $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$, all other ε_{ijk} equal zero. For simplicity, the deformations of the medium are assumed small. (What follows can be generalized to finite deformations as well.)

All functions occurring in these equations are assumed to be continuously differentiable the requisite number of times, with the exception of singular points, singular lines and singular surfaces on which these equations become meaningless. By Σ we denote a certain surface in the $x_1x_2x_3$-space. We consider the following integrals over surface Σ:

Γ-integrals of the first kind

$$\Gamma_k = \int_\Sigma \left[\left(\text{Э} + F + \frac{1}{2}\,\delta u_i^{\cdot} u_i^{\cdot}\right) n_k + (D_i E_k + B_i H_k - \sigma_{ij} u_{j,k} - q_{i,k})\, n_i\right] d\Sigma \tag{1.5}$$

Γ-integrals of the second kind

$$\Gamma_{kl} = \int_\Sigma \left[\left(\text{Э} + F + \frac{1}{2}\,\delta u_i^{\cdot} u_i^{\cdot}\right)_{,l} n_k + (D_i E_k + B_i H_k - \sigma_{ij} u_{j,k} - q_{i,k})_{,l}\, n_i\right] d\Sigma \tag{1.6}$$

Γ-integrals of the third kind

$$\Gamma_{klm} = \int_\Sigma \left[\left(\text{Э} + F + \frac{1}{2}\,\delta u_i^{\cdot} u_i^{\cdot}\right)_{,lm} n_k + (D_i E_k + B_i H_k - \sigma_{ij} u_{j,k} - q_{i,k})_{,lm}\, n_i\right] d\Sigma \tag{1.7}$$

etc. Here Э and F denote the following potentials: $\text{Э} = U - E_i D_i - H_i B_i$

$$F = -\int \left[\frac{\partial P_i}{\partial t} + \rho E_i + \varepsilon_{ijk} J_j B_k\right] dx_i \tag{1.8}$$

$$\left(\operatorname{grad} F = -\frac{\partial \mathbf{P}}{\partial t} - \rho \mathbf{E} - \mathbf{J} \times \mathbf{B}\right)$$

$$P_i = \varepsilon_{ijk} B_j D_k \qquad (\mathbf{P} = \mathbf{B} \times \mathbf{D})$$

In formulas (1.5)-(1.8) the charge density ρ is taken as constant. We adopt the usual rules of index notation for summation and differentiation (for example, $u_i^{\cdot} u_i^{\cdot} = u_1^{\cdot 2} + u_2^{\cdot 2} + u_3^{\cdot 2}$, $u_{j,k} = \partial u_j / \partial x_k$, etc.). It can be shown that the equality

$$\operatorname{rot}\left(\frac{\partial \mathbf{P}}{\partial t} + \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}\right) = 0$$

holds on the basis of Eqs. (2.1) if ρ = const. The following theorem holds being an analog of the Cauchy theorem for the given physical system.

Theorem 1.1. If: a) the surface Σ is closed; b) all functions involved in Eqs. (2.1)-(2.4) are differentiable everywhere in a domain V, surrounded by contour Σ; c) ρ = const and δ = const everywhere in domain V, then the Γ-integrals of any kind equal zero.

We prove this theorem at first for Γ-integrals of the first kind. We transform the surface integral (1.5) into a volume integral

(1.9)

$$\Gamma_k = \int_V \Big[\Big(\partial + F + \frac{1}{2}\delta u_i^{\cdot} u_i^{\cdot}\Big)_{,k} + (D_i E_k + B_i H_k - \sigma_{ij} u_{j,k} - q_{i,k})_{,i}\Big] dv$$

It can be shown that the following equation

(1.10)

$$\int_V [U_{,k} - q_{i,ki} - E_i D_{i,k} - H_i B_{i,k} - \sigma_{ij}\varepsilon_{ij,k}]\, dv = 0$$

stems from the local energy conservation law (1.4). Now we transform the integrand in formula (1.9) using formulas (1.1)-(1.3), (1.8) and (1.10). We obtain

(1.11)

$$\begin{aligned}
&\Big(\partial + F + \frac{1}{2}\delta u_i^{\cdot} u_i^{\cdot}\Big)_{,k} + (D_i E_k + B_i H_k - \sigma_{ij} u_{j,k} - q_{i,k})_{,i} = \\
&D_i (E_{k,i} - E_{i,k}) + B_i (H_{k,i} - H_{i,k}) - \varepsilon_{kij}\frac{\partial}{\partial t}(B_i D_j) - \\
&\varepsilon_{kij} J_i B_j + \delta u_i^{\cdot} u_{i,k}^{\cdot} - (\sigma_{ij} u_{i,k})_{,j} + \sigma_{ij}\varepsilon_{ij,k} = \\
&[D_i (E_{k,j} - E_{i,k}) + \varepsilon_{kij} D_j \varepsilon_{imn} E_{m,n}] + \\
&[B_i (H_{k,i} - H_{i,k}) - \varepsilon_{kij} B_i \varepsilon_{jmn} H_{m,n}] + \\
&\varepsilon_{kij}(B_i J_j - B_j J_i) + u_{i,k}(\delta u_i^{\cdot\cdot} - \sigma_{ij,j}) = 0
\end{aligned}$$

Here in the transformations we allowed for the following identity

$$\varepsilon_{kij}\varepsilon_{imn} = \delta_{kn}\delta_{jm} - \delta_{km}\delta_{jn}$$
$$(\delta_{jn} = 1 \text{ при } j = n;\ \delta_{jn} = 0 \text{ при } j \neq n)$$

and for the skew-symmetry of ε_{kij}.

The theorem has been proved for Γ-integrals of the first kind. Obviously, from the proof presented it also follows that it is valid for Γ-integrals of any kind, since their integrands in the transformed volume integrals are certain derivatives with respect to the coordinates of expression (1.11). The following theorems on the invariance of Γ-integrals ensue immediately from the theorem proved.

Theorem 1.2. Γ-integrals do not change their values along any closed surface Σ, surrounding a singular point, a singular line or a singular surface. The surface Σ can be arbitrarily deformed without changing the values of the Γ-integrals, if in the deformation the surface Σ does not intersect a singular point, a singular line or a singular surface.

Theorem 1.3. If a nonclosed surface Σ is bounded by a spatial contour L, then the Γ-integrals do not change their values under any deformation of the surface Σ if: a) contour L is fixed, b) in the deformation the surface Σ does not intersect a singular point, a singular line or a singular surface.

Strictly speaking, the Theorems 1.1-1.3 and their proof are valid only for reversible, quasistatic (nondissipative) systems, as well as(for $k = l = m = \cdots = 1$) for steady-state processes in the moving coordinate system $x'_r = x_r - Vt\delta_i{}^1$ for any arbitrary nonreversible (dissipative) system. However the theorem can be applied in other cases with some additional assumptions (for details see Cherepanov, G. P., Mechanics of Brittle Fracture, McGraw-Hill Inc., N. Y., 1978).

The results obtained earlier (see [1]) for a deformable medium in the absence of an electromagnetic field are derived from this as obvious special cases when $\mathbf{E} = 0$, $\mathbf{D} = 0$, $\mathbf{H} = 0$, $\mathbf{B} = 0$.

2. General theory of motion of singularities. Let us consider an isolated singular point O inside some domain V: by definition, all the unknown functions are differentiable everywhere in the domain V, except at the point O, at least some of them become infinite. Being infinite has no physical meaning and indicates that the mathematical theory describing the behavior of the given physical system is inaccurate. Figuratively speaking, all the errors and deviations of this theory from reality are concentrated at the singular points. The value of the integral Γ_i does not depend upon the choice of the closed contour Σ (if it only surrounds point O and lies in domain V). In particular as Σ we can take a sphere of arbitrarily small radius with center at point O.

Let us assume for $t \geqslant t_0$ the singular point O starts to move in space with velocity $\mathbf{v}$ (the s tate of point O up to the instant $t = t_0$ is of no significance). We select a moving system of Cartesian coordinates $x_1'x_2'x_3'$ with center at point O. In the moving coordinates all Γ-integrals preserve their form if we make the change $x_i \to x_i'$ and $u_i^{\cdot} \to u_i^{\cdot} - v_i$; all the theorems in Sect. 1 remain valid for the surface Σ (a small sphere, for instance) in the moving space $Ox_1'x_2'x_3'$. As the singular point moves by the distance $d\mathbf{r} = \mathbf{v}dt$, the external field (described within the framework of the given theory) does the work dA. The magnitude of this work is connected with Γ-integrals of the first kind in the following way:

$$dA = \Gamma_i dx_i \quad (dx_i = v_i dt) \tag{2.1}$$

or $\Gamma_i = \partial A/\partial x_i$. The quantity Γ_i has the dimension of a force. Thus, the physical meaning of a Γ -integral of the first kind is the following: the magnitude of Γ_i equals the irreversible work of the external field under the motion of a singular point by a unit of length along x_i.

The energy dissipation rate at the singular point is

$$A^{\cdot} = \Gamma_i v_i \tag{2.2}$$

The question is, for what values $\Gamma_i^{\cdot}$ of the energy flux and in which directions does the motion of the singular point commence? We answer this question using the tools of thermodynamics. We apply two basic approaches.

A. We assume that in the three-dimensional space $(\Gamma_1, \Gamma_2, \Gamma_3)$ there exists a surface $S(\Gamma_1, \Gamma_2, \Gamma_3) = 0$, separating the whole space into two domains: an interior one and an exterior one. If a point $(\Gamma_1, \Gamma_2, \Gamma_3)$ is in the interior domain, then the singular point in the physical space does not move; as soon as a point $(\Gamma_1, \Gamma_2, \Gamma_3)$ passes onto the surface S, the motion of the singular point in the physical space begins; the exterior domain is inaccessible. In this case the velocity of the singular point is determined from the maximum principle for the energy dissipation rate, which leads to the following expression:

$$v_i = \lambda\, \partial S/\partial\Gamma_i \tag{2.3}$$

where λ is some unknown function. Expression (2.3) determines the direction of the motion of the singular point.

This version of the construction of the theory is analogous to the theory of ideal plasticity; formula (2.3) is analogous to the associated flow rule. In case $\Gamma_2 = \Gamma_3 = 0$ with $\Gamma_1 < \Gamma_{1c}$ ·, the singular point is stationary; it starts its motion when $\Gamma_1 = \Gamma_{1c}$.

B. We assume that a dissipation function $D(\Gamma_1, \Gamma_2, \Gamma_3)$, exists, being a homogeneous function of first degree of its arguments. In this case, in accord with (2.2) we have

$$v_i = \partial D/\partial\Gamma_i \tag{2.4}$$

This version of the construction of the theory is analogous to the theory of nonlinearly-viscous bodies. In this case the motion of the singular point takes place for arbitrary values of $\Gamma_1, \Gamma_2, \Gamma_3$.

More complex synthetic models are possible for the motion of the singular point in physical space, combining the limit state and the viscous flow. The functions S and D are subject to determination from experimental data, or else from structural theory revealing the nature of the singular point. In the simplest and frequently encountered case when $\Gamma_2 = \Gamma_3 = 0$, and $v_2 = v_3 = 0$, we obviously have

$$v_1 = f(\Gamma_1) \tag{2.5}$$

Here the function $f(\Gamma_1)$ is determined from experiment or else from structural physical theory.

Let us now consider some point O on a singular line. We select a local coordinate system with center at point O, and we direct the x_3 -axis along the line. For $t > t_0$ let the singular line begin to move in some neighbourhood of point O in space with velocity $\mathbf{v}(v_1, v_2, 0)$. Having taken as Σ some cylindrical surface whose axis is the singular line, we can show that formulas (2.1) and (2.2) are valid in the case given for $i = 1, 2,$; also dA and Γ_i are line densities of the corresponding quantities at point O (i.e., are referred to a unit length of the singular line).

The dimension of Γ_1 is that of force divided by length. To determine the motion of the singular line, we can bring in the same model as before (in particular, model (2.3)-(2.5) for the corresponding cases with $i = 1, 2$).

Finally, we consider a singular surface (usually this is a surface of discontinuity on which at least some of the unknown functions undergo a jump). We select a local coordinate system with center at some point O on the surface of discontinuity and we direct the x_1 -axis along the normal to the surface. For $t > t_0$ let the singular surface move in some neighbourhood of point O in space with normal velocity v_1. The formula $dA = \Gamma_1 dx_1$, is valid here, while the quantities dA and Γ_1 are surface densities at the point O (i.e., are referred to a unit area of the singular surface). The dimension of Γ_1 is that of force divided by length squared. The general theory of motion of a singular surface is completely laid out in formula (2.5); in the simplest case we can take $\Gamma_1 = D$, where D is some constant of the material (as in explosion theory, see [1]).

Invariant Γ -integrals yield a powerful technique for the construction of new physical theories and for the unification of old ones.

3. Electrical breakdown. Let us consider an electric field in vacuum or in a dielectric. In this case

$$\mathbf{J} = 0,\ \rho = 0,\ q_i = 0,\ D_i = \varepsilon_0 E_i,\ B_i = 0,\ H_i = 0,\ u_i = 0 \tag{3.1}$$

where ε_0 is the dielectric constant of vacuum or of the dielectric (depending on the system of units). According to (1.5) the invariant Γ -integrals of the first kind in this case are

$$\Gamma_k = \frac{\varepsilon_0}{2} \int_\Sigma (- E_i E_i n_k + 2 E_i E_k n_i)\, d\Sigma \tag{3.2}$$

Maxwell's equations reduce to

$$E_i = -\varphi_{,i}, \quad \varphi_{,ii} = 0 \tag{3.3}$$

Let us consider the fundamental forms of singularities. Everywhere in what follows, as a rule, the cumbersome calculations of the invariant Γ -integrals are omitted and only the final result is presented.

Point charge in the field. The field in the neighbourhood of a point charge at the origin has the form

$$\varphi = \frac{e}{4\pi\varepsilon_0 r} - E_{i0} x_i, \quad E_i = \frac{e x_i}{4\pi\varepsilon_0 r^3} + E_{i0} \quad (r^2 = x_i x_i) \tag{3.4}$$

Here the first term is the self-field of the charge and the second term is the external field (the succeeding terms of the expansion are unessential). In this case

$$\Gamma_k = e E_{k0} \tag{3.5}$$

Thus, here the Γ_k are the components of the force with which the external field acts on the charge (if this charge is displaced by a unit of length along x_k, the energy Γ_k is dissipated).

Dipole in an external field. Let the field have the form

$$\varphi = \frac{m_i x_i}{4\pi\varepsilon_0 r^3} - E_{i0} x_i - d_{ik} x_i x_k \tag{3.6}$$

Here m_i are the components of the dipole moment and E_{i0} and d_{ik} are certain constants; the first term is the self-field of the dipole and the second and third terms represent the external field (the succeeding terms of the expansion are unessential). In this case we have

$$\Gamma_k = 2m_i d_{ik} \tag{3.7}$$

Consequently, the components Γ_k of the force with which the external field acts on the dipole are determined by the dipole moment and the derivatives of the strength of the external field at the origin. In the derivation of (3.7) we made essential use of the assumption that the characteristic structural length of the dipole is small in comparison with the radius r of sphere Σ. If the dipole is displaced by a unit of length along x_k the energy Γ_k is dissipated. A multipole of any order can be analyzed completely analogously.

Towards a theory of lightning (of discharge). Let some probe (which is in the form of a thin dielectric rod) carry a charge e at its endpoint. We place the endpoint of the rod in an external field of strength $\mathbf{E}$ (in the absence of charge e). The question is: under what conditions does a discharge (lightning) occur, i.e., the charge leaves the probe? On the basis of the general theory of motion of singularities (Sect. 2) the answer to this question is the following: when $e|\mathbf{E}| < \Gamma_c$, a discharge does not occur, while the inequality $e|\mathbf{E}| > \Gamma_c$ is impossible, i.e., the condition for the discharge is

$$e^2 (E_1^2 + E_2^2 + E_3^2) = \Gamma_c^2 \tag{3.8}$$

Here Γ_c is some local constant of the endpoint probe, characterizing the electrical-strength properties of the surrounding medium, of the probe and of their contact, as well as the geometry of the endpoint probe. For a given probe and a given external medium the constant Γ_c can be determined from one experiment of the discharge with a given charge in a given field $\mathbf{E}$. In deriving condition (3.8) from (2.3), we assumed that the external medium in the neighborhood of the endpoint probe is isotopic with respect to its resistance to the discharge, i.e., all directions are equivalent. From this assumption it follows that the limit state surface $S(\Gamma_1, \Gamma_2, \Gamma_3)$ in (2.3) is a sphere. (This argument relates also to the general case of the singular points of the field (Sect. 2).) Hence follows (3.8). In accord with (2.3), the direction of the discharge in this case coincides with the direction of the vector $\mathbf{E}$. The velocity of motion of spherical lightning is negligibly small in comparison with the velocity of line lightning. Therefore, we can assume that spherical lightning is some multipole whose structure is as yet unknown.

Line charge. Suppose that a charge is distributed uniformly along the line $x_1 = x_2 = 0$ with density q. The field in the neighborhood of this line has the form

$$E_i = \frac{q x_i}{2\pi\varepsilon_0 r^2} + E_{i0} \quad (i = 1, 2; r^2 = x_i x_i)$$

In this case

$$\Gamma_k = qE_{k0} \quad (k = 1, 2)$$

Consequently, the Γ_k again are the components of the force with which the external field acts on a unit of length of the line charge.

Vortex line. The field close to the vortex $x_1 = x_2 = 0$ (solenoidal coil) has the form

$$E_1 = -\frac{wx_2}{2\pi r^2} + E_{10}, \quad E_2 = \frac{wx_1}{2\pi r^2} + E_{20}$$
$$(\mathbf{E}_0 = \mathbf{i}E_{10} + \mathbf{j}E_{20}, \operatorname{tg} \varphi = x_2/x_1, r^2 = x_i x_i, i = 1, 2)$$

Here w is the circulation of vector $\mathbf{E}$ of the field ($\mathbf{w} = w\mathbf{k}$ is the Vector of the vortex). In this case

$$\boldsymbol{\Gamma} = \Gamma_1 \mathbf{i} + \Gamma_2 \mathbf{j} = \varepsilon_0 (\mathbf{w} \times \mathbf{E}_0) \qquad (3.9)$$

Thus, a force $\boldsymbol{\Gamma}$, defined by the formula (3.9) acts on the vortex line. A vortex line corresponds to the presence of a concentrated magnetic flux $wt\mathbf{k}$ of vector $\mathbf{B}$ along the x_3 -axis (in the core of the coil). If in the case of a line charge with $x_1 = x_2 = 0$, considered above, the observer is moving along the x_3 -axis at a constant velocity v , then he sees only the magnetic field

$$H_1 = -\frac{Jx_2}{2\pi r^2} + H_{10}, \quad H_2 = \frac{Jx_1}{2\pi r^2} + H_{20}$$
$$\left(\mathbf{J} = \frac{q}{S_0} v\mathbf{k}, J = \frac{q}{S_0} v, \mathbf{H}_0 = H_{10}\mathbf{i} + H_{20}\mathbf{j}\right)$$

where S_0 is the area of the cross section of the line charge. In this case

$$\boldsymbol{\Gamma} = \Gamma_1 \mathbf{i} + \Gamma_2 \mathbf{j} = \mu_0 (\mathbf{J} \times \mathbf{H}_0) \qquad (3.10)$$

where $\mathbf{H}_0$ is the external field of vector $\mathbf{H}$ on the x_3 -axis in the absence of current $\mathbf{J}$, and μ_0 is the magnetic permeability of the medium. Formula (3.10) is found from the expression for Γ_k in a magnetic field; this expression can be obtained from formula (3.2) by replacing E_i by H_i and ε_0 by μ_0 (where $B_i = \mu_0 H_i$).

All the results, obtained in Sect. 3, regarding the computation of forces and energy dissipation for the singularities with the aid of invariant Γ -integrals generalize easily to bodies of arbitrary form and connectivity if their typical linear dimensions are small in comparison with the radius of the sphere or cylinder Σ. Otherwise, it is necessary to apply the general formula for Γ -integrals (for example, formula (3.2) in an electrostatic field), where the surface of the body can be taken as Σ .

Lifting force in an electrostatic field. Let us consider a homogeneous electrostatic field in a plane condensor. In this field let there be a cylindrical body with a cross-section of the type of a Joukowski profile or a turbine blade (the chord of the profile is inclined at some small angle to the direction of the tension vector). We assume that the body material is an ideal insulator. Then around the profile a vortex field of vector $\mathbf{E}$ arises and a lifting force $\boldsymbol{\Gamma}$, determined by formula (3.9) acts on the profile (see Fig. 1). The magnitude of the curl w can be determined from the condition of boundedness of vector $\mathbf{E}$ at the sharp edge of the profile. This effect obtains as well for nonideal dielectrics (however, to a lesser extent). In principle, it can be used for a direct transformation of the electrostatic energy of the field into mechanical energy energy of rotation of a turbine, if we use a lattice (grid) of profile-shaped blades (as in a hydroturbine) in a sufficiently strong field $\mathbf{E}$. A lifting force of the same origin can act on uncharged oblate particle in an electric or magnetic field.

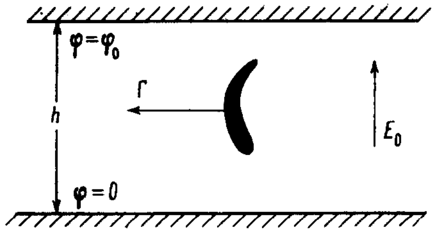

Fig. 1

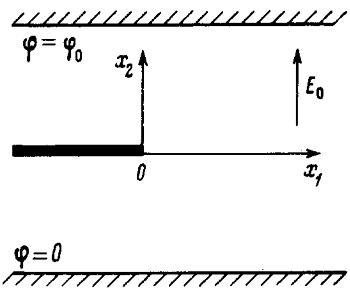

Fig. 2

Thin dielectric sheet in an electrostatic field. Let a thin plane sheet made of an ideal dielectric be placed in a plane condensor parallel to its plates. In this case the edge of the sheet is a singular line of the field **E**. We introduce the Cartesian coordinates $x_1 x_2$ with origin at some point O of the edge of the sheet where x_2 is normal to the sheet and x_1 is normal to its edge contour (Fig. 2). The field close to the edge of the sheet is

$$E_1 = -\frac{K}{\sqrt{2\pi r}}\sin\frac{\varphi}{2}, \quad E_2 = \frac{K}{\sqrt{2\pi r}}\cos\frac{\varphi}{2}$$

$$(r^2 = x_i x_i,\ \mathrm{tg}\,\varphi = x_2 / x_1,\ i = 1, 2)$$

Here K is the field intensity coefficient at the given point O; it is determined from the solution of the problem as a whole. In this case we find

$$\Gamma_1 = -{}^1/{}_2\varepsilon_0 K^2, \quad \Gamma_2 = 0$$

Consequently, a concentrated longitudinal tensile force of intensity Γ_1 is applied to the edge of the dielectric sheet (as though force lines of the field were attracting the edge of the sheet to themselves). In accord with Sect. 2 the collapse of the sheet starts at some critical value of coefficient K. The field energy decreases as the edge of the sheet moves in the condensor.

All the models constructed can be used for a theoretical description of diverse phenomena of breakdown in strong electric and magnetic fields (discharge of condensors, of the insulation in conductors, coils, etc.). The methods used for computing the forces acting on charged bodies, close to the method of Maxwell were already developed in the past century. Subsequently, this method was forgotten (for instance, it is not presented in [2]), while a simplephysical formalism, based on a representation of interaction energy, came to be applied for computing the forces. This formalism consists in the rejection of the infinite self-energy of a point source when calculating the energy of the system (the force is defined as the gradient of the energy of the system). The paradox of the divergence of the energy (*) under such an approach is logically insuperable. As we see the application of invariant Γ -integrals enables us, in particular, logically and correctly to resolve the paradox of the divergence of the energy and to give a rigorous justification of the formalism indicated.

4. Invariant Γ-integrals in hydrodynamics. We limit ourselves to the analysis of steady-state flow of an ideal incompressible weightless liquid, described by the equations

$$\varphi_{,ii} = 0, \quad v_i = \varphi_{,i}, \quad p = p_0 - {}^1/_2 \delta v_i v_i \quad (i = 1, 2, 3) \tag{4.1}$$

Here φ, v_i and p are the flow potential, the velocity components and the pressure in the liquid, and p_0 is the pressure at a critical point of the flow. In this case the invariant Γ -integrals are

$$\Gamma_k = \frac{1}{2} \delta \int_\Sigma (-v_i v_i n_k + 2 v_i n_i v_k) \, d\Sigma \tag{4.2}$$

$$\Gamma_{kl} = \frac{1}{2} \delta \int_\Sigma [-(v_i v_i)_{,l} n_k + 2 (v_i v_k)_{,l} n_i] \, d\Sigma \tag{4.3}$$

etc. It is not difficult to see that the quantities Γ_k equal the corresponding components of the force acting on a body placed inside a closed surface Σ in the flow of the liquid (for undetached flow).

Let us consider some examples.

Source in liquid flow. Let the flow field be formed by superimposing a source field q_v on the unperturbed flow

$$v_i = \frac{q_v x_i}{4\pi r^3} + v_{i0} \quad (r^2 = x_i x_i, \quad i = 1, 2, 3)$$

In this case we have

$$\Gamma_k = -\delta q_v v_{k0} \; (\varkappa = 1, 2, 3)$$

Thus, a source moves against the flow and a sink with the flow.

Dipole in liquid flow. Let the flow field have the following form:

$$\varphi = \frac{m_i x_i}{4\pi r^3} - v_{i0} x_i - d_{ik} x_i x_k$$

Here the first term is the self-field of the dipole with the moment (m_1, m_2, m_3), and

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*) To resolve this paradox for an electron, Landau and Lifshits introduce the assumption of an infinite negative mass of an electron of nonelectromagnetic origin (see p. 123 in [2]).

the second and third terms represent the unperturbed external field. In this case we have

$$\Gamma_k = 2\,\delta\, m_i d_{ik} \tag{4.4}$$

Thus, the components Γ_k of the force with which the external field acts on the dipole are determined by the dipole moment and the gradient of the velocity of the unperturbed flow.

Flow past a body. The self-field of a body in a flow at distances large in comparison with its size, is a dipole in the case of undetached flow. Therefore, by deforming the contour Σ, we arrive at formula (4.4) for the resultant forces acting on the body in the flow. In particular, if the unperturbed flow is homogeneous, i.e., $d_{ik} = 0$, we obtain $\Gamma_k = 0$ (the d'Alemebert-Euler paradox). In models with a reverse flow (of the type of the Efros-Gilbarg-Ross model) the body experiences a drag which can be found with the aid of Γ-integrals. By deforming the closed surface Σ in integral (4.2) from a sphere of infinite radius into the frontal surface of the body plus the surface of the cavity, we obtain the following expression for the drag F of an axisymmetric body under a stalled axisymmetric cavitational flow with reverse flow:

$$F = \delta v_\infty^2 \frac{S_J}{S_f}\left(1 + Q + \sqrt{1+Q}\right) \qquad \left(Q = \frac{p_\infty - p_c}{{}^1/_2\delta v_\infty^2}\right)$$

Here p_∞ and v_∞ are the pressure and the velocity of the unperturbed flow, p_c is the pressure in the cavity, S_J is the asymptotic cross section of the reverse flow and S_f is the frontal section of the body. With the aid of invariant Γ-integrals we can also obtain a number of known classical results (for example, the Levi-Civita formula for the flow past a body with an infinite cavity, the thickness of the jet in a Borda orifice, the Cisotti effect, etc.).

Vortex in a liquid flow. Let the flow field have the form

$$v_1 = -\frac{wx_2}{2\pi r^2} + v_{10}, \qquad v_2 = \frac{wx_1}{2\pi r^2} + v_{20}, \qquad v_3 = 0$$
$$(r^2 = x_i x_i,\ i = 1,2)$$

Here the first term is the self-field of the vortex $x_1 = x_2 = 0$, the second term is the unperturbed flow and w is the circulation of the vector $\mathbf{v}$. In this case

$$\boldsymbol{\Gamma} = \Gamma_1\mathbf{i} + \Gamma_2\mathbf{j} = \delta w\,(\mathbf{k}\times\mathbf{v}_0) \tag{4.5}$$
$$(\mathbf{v}_0 = v_{10}\mathbf{i} + v_{20}\mathbf{j})$$

By deforming the contour Σ in the Γ-integral (4.2), we obtain the following result: under a circulational flow past a cylindrical body of arbitrary section, the lifting force is determined by formula (4.5) (Joukowski's theorem).

5. Theory of cracks and dislocations in elastic bodies. The invariant Γ-integrals of the nonlinear theory of elasticity have the form

$$\Gamma_k = \int_\Sigma \left[\left(U + \frac{1}{2} \delta u_i^{\cdot} u_i^{\cdot} \right) n_k - \sigma_{ij} u_{i,k} n_j \right] d\Sigma \tag{5.1}$$

$$\Gamma_{kl} = \int_\Sigma \left[\left(U + \frac{1}{2} \delta u_i^{\cdot} u_i^{\cdot} \right)_{,l} n_k - (\sigma_{ij} u_{i,k})_{,l} n_j \right] d\Sigma \tag{5.2}$$

etc. Let us consider the basic types of singularities of an elastic field.

Concentrated force on the free boundary of a halfspace. Let a homogeneous isotropic linearly elastic halfspace $x_3 \geqslant 0$ be subjected to a concentrated force $(P_1, 0, 0)$, applied at the origin. The elastic field in the neighborhood of the origin has the following form:

$$\begin{aligned} 4\pi\mu u_1 &= P_1 \left[\frac{1}{r} + \frac{x_1^2}{r^3} + \frac{1-2\nu}{r+x_3} - \frac{x_1^2(1-2\nu)}{r(r+x_3)^2} \right] + a_{1k} x_k \\ 4\pi\mu u_2 &= P_1 \frac{x_1 x_2}{r} \left[\frac{1}{r^2} - \frac{1-2\nu}{(r+x_3)^2} \right] + a_{2k} x_k \\ 4\pi\mu u_3 &= P_1 \frac{x_1}{r} \left(\frac{x_3}{r^2} + \frac{1-2\nu}{r+x_3} \right) + a_{3k} x_k \quad (r^2 = x_i x_i) \end{aligned} \tag{5.3}$$

Here (u_1, u_2, u_3) is the displacement vector, μ is the shear modulus, ν is the Poisson's ratio, a_{ik} are constants determining the unperturbed external field. The first term with factor P_1 is the self-field of the concentrated force. In this case we have(*)

$$\begin{aligned} \Gamma_1 &= -\frac{P_1}{24\pi\mu} \left\{ a_{11} \left[\frac{3\nu}{4} + \frac{8(1+2\nu)}{5(1-2\nu)} + \frac{5}{8} + \frac{51}{10} \right] + \right. \\ &\left. (1-\nu) a_{33} + \nu a_{22} - \frac{1}{2}(a_{12} + a_{21}) \left(\frac{3}{10} + \nu \right) \right\} \\ \Gamma_2 &= \frac{P_1 a_{12}}{8\pi\mu(1-2\nu)} \left(\frac{37}{24} - \frac{7}{20}\nu - \frac{2}{3}\nu^2 \right) \end{aligned} \tag{5.4}$$

Concentrated line force inside a body. Let the elastic body be a thin plate located in the plane $x_3 = 0$. A longitudinal concentrated force (P_1, P_2) per unit of the thickness of the plate is applied to the plate at the origin. Let the elastic field in terms of the complex Kolosov-Muskhelishvili potentials $\varphi(z)$ and $\psi(z)$ have the form

$$\begin{aligned} \varphi(z) &= -\frac{P_1 + iP_2}{2\pi(1+\varkappa)} \ln z + (A_1 + iA_2) z \\ \psi(z) &= \frac{\varkappa(P_1 - iP_2)}{2\pi(1+\varkappa)} \ln z + (B_1 + iB_2) z \\ &\left(z = x_1 + ix_2, \ \varkappa = \frac{3-\nu}{1+\nu} \right) \end{aligned} \tag{5.5}$$

*) A. S. Bykovtsev computed (5.4)

Here A_i and B_i are real constants determining the external unperturbed field; the first term is the self-field of the concentrated force. In this case we have (**)

$$\Gamma_1 = \frac{1+\nu}{E}\{[(2-4\nu)A_1 - B_1]P_1 + [(4-4\nu)A_2 + B_2]P_2\}$$
$$\Gamma_2 = \frac{1+\nu}{E}\{[(2-4\nu)A_1 + B_1]P_2 + [-4(1-\nu)A_2 + B_2]P_1\} \tag{5.6}$$

Theory of the strength of rivets. Let a thin plate be attached at the origin to another elastic body (a rod, a plate or a massive body). This joining of elastic bodies shall be called riveting for concreteness; however, we should bear in mind that the theory being proposed refers to any technological operation or method of attachment as long as the size of the rivet is small in comparison with the typical dimensions of the body. We assume that the rivet has sufficient strength so that the fracture (or limit state) under sufficiently large loads takes place close to the rivet in the plate. On the basis of the general theory in Sect. 2 the condition for the fracture is the following (here we used the condition of isotropy with respect to strength):

$$\Gamma_1^2 + \Gamma_2^2 = \Gamma_c^2 \tag{5.7}$$

where Γ_1 and Γ_2 are given by (5.6) and Γ_c is some local constant depending on the construction of the rivet and method of attachment and on the local strength of the plate. The quantity Γ_c is independent of the external field, the geometry of the plate, the loads, etc.; all these factors enter into the left-hand side of (5.7.). The value of Γ_c must be determined experimentally. An analogous theory can be developed for the fracture of the attachment to a massive body, by using the formula (5.6) and the general theory in Sect. 2.

Linear dislocation. Let an elastic body, finding itself in a state of a plane state of stress or strain, contain a dislocation along the line $x_1 = x_2 = 0$. Let the elastic field close to the origin of the plane x_1x_2 be

$$\varphi(z) = \frac{\mu(b_1 + ib_2)}{\pi i(\varkappa+1)}\ln z + (A_1 + iA_2)z$$
$$\psi(z) = -\frac{\mu(b_1 - ib_2)}{\pi i(\varkappa+1)}\ln z + (B_1 + iB_2)z \qquad (z = x_1 + ix_2) \tag{5.8}$$

Here (b_1, b_2) is the Burgers dislocation vector, u is the shear modulus; the first term is the self-field of the dislocation and the second is the unperturbed external field. In this case we have

$$\Gamma_1 = b_1B_2 + b_2(2A_1 + B_1), \quad \Gamma_2 = b_1(-2A_1 + B_1) - b_2B_2 \tag{5.9}$$

The quantitites A_1, B_1, B_2 have the following physical meaning

$$A_1 = {}^1/_4\,(\sigma_{11}^\circ + \sigma_{22}^\circ), \quad B_1 = {}^1/_2\,(\sigma_{22}^\circ - \sigma_{11}^\circ), \quad B_2 = \sigma_{12}^\circ \tag{5.10}$$

**) L. A. Kipnis computed (5.6)

Here σ_{11}°, σ_{12}° and σ_{22}° are the stresses of the unperturbed external field at the origin. Using (5.10), formula (5.9) can be written as

$$\Gamma_i = \varepsilon_{ij3}\sigma_{jk}^\circ b_k \tag{5.11}$$

This is the well-known Peach-Kochler formula for the configurational force acting on a dislocation. Dislocation theory is based on it.

In accord with the general theory in Sect. 2, the motion of a dislocation takes place as soon as the absolute value of vector $\boldsymbol{\Gamma}$, equal to $\sqrt{\Gamma_i\Gamma_i}$, reaches a critical value Γ_c, lying on the limit polar S (Γ_1, Γ_2) (which is found from experiment). In case of isotropy the polar is the circumference of a circle and the magnitude of Γ_c does not depend upon the direction of motion of the dislocation. However, if the direction of the motion of the dislocation is known from experiment, then the magnitude of Γ_c equals the dissipation of the field energy as the dislocation moves in the direction indicated by a unit of length (this magnitude is determined by experiment or from structure theory). The kinetics of the motion of dislocation in time under a constant external load must be described within the framework of the dependence of the vector $\mathbf{v}$ on the vector $\boldsymbol{\Gamma}$ (see Sect. 2).

Cleavage cracks. Let the front of a crack in an elastic body coincide with the line $x_1 = x_2 = 0$. The edges of the crack along $x_2 = 0$ and $x_1 < 0$ are free of external loads (cleavage crack). The conditions of plane strain or of plane state of stress are assumed. Let an elastic field close to the crack front have, in the complex potentials $\Phi(z)$ and $\Omega(z)$, the form

$$2\Phi(z) + \Omega(z) = \frac{K_{\mathrm{I}}}{\sqrt{2\pi z}}, \quad \Omega(z) = \frac{iK_{\mathrm{II}}}{\sqrt{2\pi z}} \quad (z = x_1 + ix_2) \tag{L}$$

Here K_{I} and K_{II} are stress intensity factors. In this case

$$\boldsymbol{\Gamma} = -\frac{\varkappa + 1}{8\mu}\left[(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2)\mathbf{i} - 2K_{\mathrm{I}}K_{\mathrm{II}}\mathbf{j}\right] \tag{M}$$

The theory of crack development is obtained from the general theory of motion of singularities (Sect. 2) as a special case. Invariant Γ-integrals have been used also for describing the motion of a singular fracture surface in the theory of the effect of explosions in brittle bodies [1].

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